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Diffraction from an Irregular Screen with Applications to Ionospheric Problems

H. G. Booker, J. A. Ratcliffe and D. H. Shinn

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DIFFRACTION FROM AN IRREGULAR SCREEN WITH APPLICATIONS TO IONOSPHERIC PROBLEMS

By H. G. BOOKER, J. A. RATCLIFFE and D. H. SHINN
Cavendish Laboratory, University of Cambridge

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An analysis is made of the diffraction effects produced when a plane wave is incident upon an irregular diffracting screen, and the results are applied to the problem of the reflexion of radio waves from an ionosphere which is irregular in the horizontal plane. The nature of the irregular screen is assumed to be given in terms of the variation of electric wave-field in a plane just beyond the screen, and it is assumed that variations occur over the plane in one direction only. It is further assumed that the screen is 'random' in the sense that it is one of an assembly all of which differ from each other, but have statistical properties in common, and deductions are made about the diffraction patterns averaged over the assembly. It is shown that many aspects of the problem can be investigated by use of the theory of 'random' electrical noise as developed by Rice and Uhlenbeck. The angular spectrum (Fraunhofer diffraction pattern) and the Fresnel diffraction pattern are described in terms of their spatial auto-correlation functions, and there is some discussion of a related method of dealing with Fresnel diffraction problems from completely determined screens.

In part II of the paper the irregular 'fading' exhibited by a radio wave returned from the ionosphere is discussed in terms of two models in which the fading is assumed to be produced by movements of the diffracting centres in the ionosphere. The temporal auto-correlation function of the amplitude of the irregularly fading signal is related to the velocity of the ionospheric diffracting centres.

1. INTRODUCTION

When a radio wave is returned to the ground after a single reflexion at vertical incidence from the ionosphere it is found that its amplitude alters more or less rapidly and it is said to 'fade'. Although this 'fading' is found to be closely correlated at two receiving points situated close together, the correlation becomes less good as the separation of the receivers is increased beyond one wave-length, and there is little correlation for considerably larger separations (Ratcliffe & Pawsey 1933; Pawsey 1935; Ratcliffe 1948). Simple arguments, such as those given by Ratcliffe & Pawsey (1933), lead immediately to the conclusion that the ionosphere must be non-uniform in the horizontal plane, and what is usually thought of as a single reflected 'ray' must be in reality a rather wide cone of waves scattered back to the receiving point from the irregularities. Once this picture has been accepted it is clear that a detailed investigation of the problem is best made by considering the diffraction of waves incident on the ionosphere.

In this paper the problem is divided into two parts. In the first an investigation is made of the distribution of wave-field over the ground, due to a wave which has been diffractively reflected from an irregular ionosphere of constant form. In the second the analysis is extended to deal with an irregular ionosphere whose structure varies with time, and the resulting temporal variations (fading) of the wave-field at the ground are discussed.

Where it is convenient the actual case of an irregularly reflecting ionosphere, irradiated from a 'point-source' transmitter, has been replaced by a simplified model consisting of an irregular transmitting screen illuminated by a parallel beam of radiation, so that the reflexion problem is replaced by a transmission problem. Important characteristics of the transmitted radiation are deduced in terms of the statistical properties of the wave-field just after it leaves the screen. No consideration is given to the type of ionosphere which would produce this irregular wave-field just below itself. The results deduced in this way have an application to several optical problems of diffraction from irregular screens.

PART I. DIFFRACTION FROM A RANDOM SCREEN OF CONSTANT FORM

2. Outline of the problem

The existing experimental evidence indicates that the distribution of the field over the ground is irregular, and changes from time to time in a random manner. If we had complete knowledge of the distribution at any one instant, then, ideally, it would be possible to make deductions about the ionosphere at that instant, but a short time later the ionospheric structure would have changed, and with it the details of the diffraction pattern at the ground. We are not so much interested in the precise structure at any one instant as in those statistical properties which are common to the structures at successive times.

The problem is similar to that encountered in a study of 'radio noise' such as arises from a resistor kept at a fixed temperature. In this case the resistor gives rise to an irregular e.m.f. which can be completely described, after it has occurred, either in terms of its time-variation or in terms of its 'frequency-spectrum', but the e.m.f. produced during a subsequent period of time will have a different time-variation and a different frequency-spectrum. Because the precise nature of the fluctuating e.m.f. cannot be described it is said to be 'random', but there are nevertheless certain statistical characteristics of the fluctuation which can be described and which are common to all the samples observed. One of the most important characteristics of the fluctuation from this point of view is its 'auto-correlation function', which indicates the average degree of correlation between two observations of e.m.f. made at times separated by a series of different intervals. Rice (1944, 1945) has shown how to specify random radio noise in terms of this function.

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In § 4 we use the analogy of radio noise to develop an analysis of diffraction in terms of an 'auto-correlation function' which measures the degree of correlation between two magnitudes observed at places separated by a given distance, and in § 5 we apply this method of analysis to the case of a random field distribution such as might be produced by the ionosphere.

In the treatment referred to above, we consider the nature of the angular spectrum of radiation which would be produced by a known or random distribution of field intensity. This angular spectrum is closely allied to the 'Fraunhofer' diffraction pattern produced by the field distribution. In §§ 6 and 7 we consider the 'Fresnel' diffraction pattern produced on a plane at a finite distance from that in which the field distribution is known. This pattern corresponds more nearly to that formed at the ground by a radio wave.

Our consideration of the radio-wave diffraction pattern formed at the ground will lead us to the conclusion that the ionosphere might sometimes behave approximately like a 'completely rough' diffractive reflector, and in § 8 we examine more fully the nature of diffraction at a 'completely rough' screen of this type.

Throughout this paper we shall be concerned only with fundamental problems of diffraction and shall not consider the nature, or cause, of the ionospheric irregularities which give rise to the field distributions we postulate.

3. The angular spectrum from a known distribution of field

We shall consider the problem of an electro-magnetic wave which has passed through a screen whose transmission properties vary across the aperture and we shall first assume that we know the magnitudes of the field components of the wave at each point just after it has passed through the screen. Then we can represent the complete wave emerging from the screen in terms of an *angular spectrum* of plane waves; the angular spectrum being related to the Fourier analysis of the wave disturbance over the aperture. This is a well-known way of calculating the 'polar-diagram' of a radio aerial, but for the sake of completeness we shall outline the analysis here. Further details are given in a paper by Booker & Clemmow (1950). We shall restrict ourselves to a two-dimensional case, in which variations in the wave-field as it emerges from the diffracting screen occur only in one direction.

Let the screen lie in the XY plane and let there be no variation of its properties in the Y-direction. Let a parallel beam of radiation of wave-length λ be incident normally on the screen from the negative Z-direction, and let the distribution of electric field after the wave has passed the screen be given by $E_x(x,0)$ and $E_y(x,0)$ for the two different components of polarization respectively. Since the component E_x varies along the X-direction it can be shown that there must also be a component $E_z(x,0)$ of field in the Z-direction. Now let the functions E_x, E_y, E_z be expressed as Fourier integrals as follows:

$$\left.\begin{aligned} E_x(x,0) &= \int_{-\infty}^{\infty} G_x(\kappa)\exp(2\pi i\kappa x)\,d\kappa, &\text{(i)}\\ E_y(x,0) &= \int_{-\infty}^{\infty} G_y(\kappa)\exp(2\pi i\kappa x)\,d\kappa, &\text{(ii)}\\ E_z(x,0) &= \int_{-\infty}^{\infty} G_z(\kappa)\exp(2\pi i\kappa x)\,d\kappa. &\text{(iii)}\end{aligned}\right\} \quad (1)$$

Next let us build up the field distribution over the XY plane from an 'angular spectrum' consisting of a series of plane waves travelling in the ZX plane and making different angles θ with OZ. Let $F_1(\theta)\,d\theta$ be the (complex) amplitude of waves polarized with electric vector in the XZ plane and with wave normals in the range of angles from θ to $\theta+d\theta$; so that

$$F_1(\theta)\exp\left(\frac{2\pi i}{\lambda}(x\sin\theta+z\cos\theta)-2\pi ift\right)d\theta$$

represents the electric field due to the wave. Similarly, let $F_2(\theta)\,d\theta$ be the corresponding quantity for waves polarized with the electric vector along OY. Then the time-varying electric field components $\mathscr{E}_x$, $\mathscr{E}_y$, $\mathscr{E}_z$ at any point (x, z) are given by the following expressions, where $s\equiv\sin\theta$ and $c\equiv\cos\theta$:

$$\left.\begin{aligned}\mathscr{E}_x(x,z,t)&=\left[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_1(\theta)\cos\theta\exp\left(\frac{2\pi i}{\lambda}sx\right)d\theta\right]\exp\left\{2\pi i\left(\frac{cz}{\lambda}-ft\right)\right\}, &\text{(i)}\\ \mathscr{E}_y(x,z,t)&=\left[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_2(\theta)\exp\left(\frac{2\pi i}{\lambda}sx\right)d\theta\right]\exp\left\{2\pi i\left(\frac{cz}{\lambda}-ft\right)\right\}, &\text{(ii)}\\ \mathscr{E}_z(x,z,t)&=\left[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_1(\theta)\sin\theta\exp\left(\frac{2\pi i}{\lambda}sx\right)d\theta\right]\exp\left\{2\pi i\left(\frac{cz}{\lambda}-ft\right)\right\}. &\text{(iii)}\end{aligned}\right\}\quad(2)$$

On the plane $z=0$ the field components $E_{x,y,z}(x,0)$, defined by

$$\mathscr{E}_{x,y,z}(x,0)=E_{x,y,z}(x,0)\exp(-2\pi ift),$$

are given by

$$\left.\begin{aligned}E_x(x,0)&=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_1(\theta)\cos\theta\exp\left(\frac{2\pi is}{\lambda}x\right)d\theta, &\text{(i)}\\ E_y(x,0)&=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_2(\theta)\exp\left(\frac{2\pi is}{\lambda}x\right)d\theta, &\text{(ii)}\\ E_z(x,0)&=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}F_1(\theta)\sin\theta\exp\left(\frac{2\pi is}{\lambda}x\right)d\theta. &\text{(iii)}\end{aligned}\right\}\quad(3)$$

We can now relate the functions $G_{x,y,z}(\kappa)$ of equations (1) with the functions $F_{1,2}(\theta)$ of equations (3) by writing

$$\kappa=\frac{s}{\lambda},\quad d\kappa=\frac{ds}{\lambda}=c\frac{d\theta}{\lambda},$$

and hence

$$\left.\begin{aligned}F_1(\theta)&=\frac{1}{\lambda}G_x\left(\frac{s}{\lambda}\right), &\text{(i)}\\ F_2(\theta)&=\frac{1}{\lambda}\cos\theta\,G_y\left(\frac{s}{\lambda}\right), &\text{(ii)}\\ F_1(\theta)&=\frac{1}{\lambda}\cot\theta\,G_z\left(\frac{s}{\lambda}\right). &\text{(iii)}\end{aligned}\right\}\quad(4)$$

In what follows we shall restrict our attention to the wave-component polarized in the x-direction, since that is the direction in which we shall consider the diffracting screen to be varying. Equations (1) (i), (3) (i) and (4) (i) then show us that $G_x(s/\lambda)$ is the Fourier transform of $E_x(x,0)$ and is also equal to $\lambda F_1(\theta)$ which is λ times the amplitude in the angular spectrum of the wave-disturbance. We therefore write $G_x(s/\lambda) = P(s)$ and call it the 'angular spectrum' of the wave disturbance. We then have from (1) (i)

$$E_x(x, 0) = \frac{1}{\lambda}\int_{-\infty}^{\infty} P(s) \exp(2\pi i s x/\lambda)\, ds, \tag{5}$$

and by the theorem of Fourier transforms

$$P(s) = \lambda\int_{-\infty}^{\infty} E_x(x, 0) \exp(-2\pi i s x/\lambda)\, dx, \tag{6}$$

and, noting that $P(s) = \lambda F_1(\theta)$, we see from equation (2) (i) that the field $E_x(x, z)$ at any point (x, z) is given by

$$E_x(x, z) = \frac{1}{\lambda}\int_{-\infty}^{\infty} P(s) \exp\left\{\frac{2\pi i}{\lambda}(sx+cz)\right\} ds, \tag{7}$$

where $c = \surd(1-s^2)$. The wave-system is thus completely described by the function $P(s)$ which we have called the angular spectrum of the wave, and $P(s)$ is given in terms of the field distribution over the plane XY by the expression (6). Equations (5), (6) and (7) show that it is convenient to express distances in terms of the wave-length λ, and we shall from now on express them in this way.

In the case of optical diffraction we can effectively observe the angular spectrum $P(s)$ by using a converging lens and observing the distribution of illumination in its focal plane. This distribution is usually called the 'Fraunhofer' diffraction pattern.

The 'Fresnel' diffraction pattern of optics is that which is produced over a plane at a finite distance z from the screen, and so it corresponds to the function $E_x(x, z)$ given by equation (7). For a known distribution of $E_x(x, 0)$ across the aperture we therefore have available a method for calculating both types of diffraction pattern. In the following sections we shall consider, in more detail, the calculation of the angular spectrum and of the Fresnel diffraction pattern, both for aperture distributions which are completely known and for those which are only specified in terms of their statistical properties.

If the screen is of limited aperture it can be proved (Booker & Clemmow 1950) that if we move from O in a direction θ_1 to a distance r_1 from the screen large compared with its extent, then the resultant electric field at the point (r_1, θ_1) is proportional to $\dfrac{P(\sin\theta_1)}{\surd r_1}$. Thus if we go far enough away from a screen of limited aperture we can observe its angular spectrum by direct measurement. Under these conditions the angular spectrum is also referred to as the 'polar diagram' of the radiating screen. Calculations based on these ideas are well known in the theory of aerials (see Fry & Goward 1949).

4. The angular spectrum in terms of the auto-correlation function

It has been explained in § 2 that it will be convenient to express the properties of a random field distribution in terms of its auto-correlation function, and therefore, as a preliminary, we here consider the calculation of the angular spectrum of a known distribution in terms of this function.

We first define a generalized auto-correlation function, which applies to complex functions, in the following way. Let $f(x)$ represent a real or complex function of x, and define its auto-correlation function $\rho(\xi)$ by the expression

$$\rho(\xi) = \frac{\int_{-\infty}^{+\infty} f^*(x)\, f(x+\xi)\, dx}{\int_{-\infty}^{+\infty} f^*(x)\, f(x)\, dx}. \tag{8}$$

Let us now transform this expression for $\rho(\xi)$ as follows. Let $f(x)$ and $g(x)$ be any two functions of x, real or complex, and let $F(s)$ and $G(s)$ be their respective Fourier transforms.

Then
$$F(s) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i x s)\, dx,$$
$$G(s) = \int_{-\infty}^{+\infty} g(x) \exp(-2\pi i x s)\, dx.$$

Therefore
$$\int_{-\infty}^{+\infty} f(x)\, g(\xi - x)\, dx = \iint f(x)\, G(s) \exp\{2\pi i s(\xi - x)\}\, dx\, ds$$
$$= \int_{-\infty}^{+\infty} F(s)\, G(s) \exp(2\pi i s \xi)\, ds. \tag{9}$$

This is the 'convolution theorem' of Fourier transforms. Now put $g(x) = f^*(-x)$, then $G(s) = F^*(s)$ and we obtain

$$\int_{-\infty}^{+\infty} f(x)\, f^*(x-\xi)\, dx = \int_{-\infty}^{+\infty} F(s)\, F^*(s) \exp(2\pi i s \xi)\, ds.$$

Now write $x+\xi$ instead of x, and note that $F(s)\, F^*(s) = |F(s)|^2$, and we obtain

$$\int_{-\infty}^{+\infty} f^*(x)\, f(x+\xi)\, dx = \int_{-\infty}^{+\infty} |F(s)|^2 \exp(2\pi i s \xi)\, ds. \tag{10}$$

Similarly,
$$\int_{-\infty}^{+\infty} F^*(s)\, F(s+\sigma)\, ds = \int_{-\infty}^{+\infty} |f(x)|^2 \exp(-2\pi i \sigma x)\, dx. \tag{11}$$

By putting $\xi = 0$ in equation (10) we obtain

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(s)|^2\, ds. \tag{12}$$

A combination of equations (8), (10) and (12) gives the following expression for the auto-correlation function of $f(x)$ in terms of its Fourier transform:

$$\rho(\xi) = \frac{\int_{-\infty}^{+\infty} |F(s)|^2 \exp(2\pi i s \xi)\, ds}{\int_{-\infty}^{+\infty} |F(s)|^2\, ds}. \tag{13}$$

We may notice in passing that if $f(x)$ is real $|F(s)|$ is a symmetrical function, and (13) becomes

$$\rho(\xi) = \frac{\int_{0}^{\infty} |F(s)|^2 \cos(2\pi s \xi)\, ds}{\int_{0}^{\infty} |F(s)|^2\, ds}. \tag{14}$$

This is Wiener's theorem (Wiener 1930), which is much used in the theory of random electrical noise. In that case $|F(s)|^2$ represents the 'power spectrum' of the noise, and the theorem is often quoted by saying that 'the auto-correlation function of a real function is proportional to the cosine Fourier transform of its power spectrum'. In this paper we shall use the more general expression (13).

We can now use the auto-correlation function $\rho_E(\xi)$ to specify the relation between the distribution $E_x(x)$ of the wave-field over a diffracting screen and the power $|P(s)|^2$ diffracted in a given direction $s = \sin\theta$. To do this write $f(x) \equiv E_x(x)$ in equation (8), and recall from (5) that the Fourier transform of $E_x(x)$ is $\frac{1}{\lambda} P(s)$, and we seen from (13) that

$$\rho_E(\xi) = \frac{\int_{-\infty}^{+\infty} |P(s)|^2 \exp(2\pi i s\xi)\, ds}{\int_{-\infty}^{+\infty} |P(s)|^2\, ds}. \tag{15}$$

This expression relates the angular power spectrum $|P(s)|^2$ to the auto-correlation function $\rho_E(\xi)$ of the wave-field $E_x(x, 0)$, whereas equation (6) relates the complex amplitude $P(s)$ of the angular spectrum to the Fourier transform of $E_x(x, 0)$.

It is interesting to note that if we are given only the distribution of power $|P(s)|^2$ in the angular spectrum, and not its phase, we can deduce only the auto-correlation function of the field producing that spectrum. But since a given auto-correlation function can result from an infinite set of different field distributions it follows that the field distribution is not uniquely determined when only the angular spectrum of power is known. An application of this principle to radio aerials shows that several aerials having different distributions of field across their apertures could all give the same 'polar diagram' of radiated power, provided the auto-correlation functions of the aperture distribution were the same.

A simple extension of equation (15) to three dimensions is of use in the problem of analyzing crystals by means of X-rays, where, as is well known, it is not possible to determine the phase of the diffracted radiation. It is at once apparent that, given only the amplitude of the angular spectrum scattered from the crystal it is possible to determine the auto-correlation function of the scattering power of the crystal. This auto-correlation function is precisely what is determined in that method of X-ray investigation which makes use of the so-called 'Patterson Series' (James 1948; Patterson 1934, 1935).

We can summarize the results of this section by stating equation (15) in words as follows: 'The auto-correlation function of $E_x(x)$ is proportional to the Fourier transform of the distribution of power $|P(s)|^2$ in the angular spectrum.'

5. The angular spectrum from a random distribution of field

(a) *A random distribution of field*

In the preceding section we have assumed that the intensity $E(x, 0)$ in the field of the wave just after it leaves the diffracting screen is completely known, and have shown that the angular power spectrum $|P(s)|^2$ can be deduced and can be related to the auto-correlation function $\rho_E(\xi)$ of $E(x, 0)$. The function $E(x, 0)$ might be quite irregular, as would occur, for example, if the screen were a piece of ground glass, but that would not influence our conclusion provided that it was completely determined. For example, in the case where $E(x, 0)$ is real but irregular the functions $\rho_E(\xi)$ and $|P(s)|^2$ might have the forms shown in figure 1. Both functions would have a regular mean trend with an irregular structure superimposed, and from equation (15) they would be Fourier transforms of each other, apart from a scaling factor.

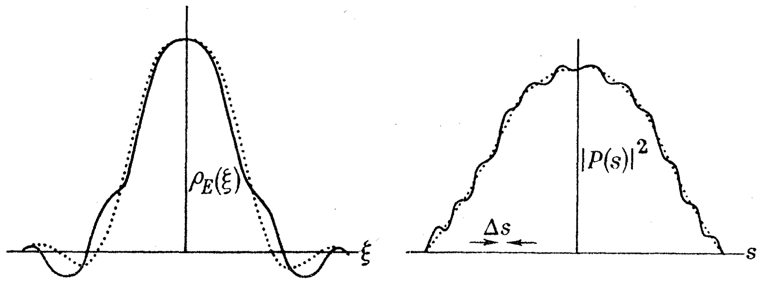

FIGURE 1. To illustrate the nature of the functions $\rho_E(\xi)$ and $|P(s)|^2$ for a known, but irregular, distribution of field $E_x(x, 0)$. The diagrams are drawn for the case where $E_x(x, 0)$ is real. The dotted lines represent averages over systems.

Suppose next that instead of one sheet of ground glass we have a large series of separate samples, all made by the same process so that, although the precise fine-structure of the corresponding wave-fields varied from each to each, their general statistical nature was the same. We shall assume that this statistical similarity implies that the auto-correlation functions $\rho_E(\xi)$ for each piece of glass separately have almost exactly the same shape.

Although the precise nature of each separate field distribution would be unknown the common statistical properties would enable us to make useful deductions about the average behaviour of a large series of samples. We shall often call averages of this kind 'system averages' or 'averages over systems', and shall denote them by a line drawn over the appropriate symbol.

We shall also be concerned with another type of average, namely, the average value of a quantity over a single aperture distribution. In nearly all cases which we shall consider these two averages will be the same. This happens only if the aperture average does not vary between one field distribution and another, and the system average of the quantity does not vary from point to point of the aperture. This implies that a statistical examination of the irregularities in the field distribution would not lead to different results for different distributions nor to different results at different points of one distribution.

Under these conditions expressions such as that of equation (8) can be written in one of two alternative forms, as follows:

$$\rho(\xi) = \frac{\int_{-\infty}^{+\infty} f^*(x)\, f(x+\xi)\, dx}{\int_{-\infty}^{+\infty} f^*(x)\, f(x)\, dx}, \tag{8}$$

or

$$\rho(\xi) = \frac{\overline{f^*(x)\, f(x+\xi)}}{\overline{f^*(x)\, f(x)}}. \tag{8a}$$

In this paper, when we say that the distribution of field over a plane is 'random' we shall mean that we are considering a large number of such distributions which have common statistical properties but differ from each other in detail; one of the statistical properties is that the auto-correlation function of the field is almost exactly the same for each distribution.

(*b*) *The average spectrum from a random distribution of field*

Let us now consider the case of a 'random' set of diffracting screens of the type discussed above, limited by an optical 'stop' and illuminated by light of constant intensity. For *one* of the system of screens $E(x, 0)$ would be fully determined and, as before, $\rho_E(\xi)$ and $|P(s)|^2$ might have the forms shown in figure 1. When we have to deal with the series of screens, however, it is more useful to consider the averages of these quantities over systems, and by averaging equation (15) over systems in this way we arrive at the result†

$$\overline{\rho_E(\xi)} = \frac{\int_{-\infty}^{+\infty} \overline{|P(s)|^2} \exp(2\pi i \xi s)\, ds}{\int_{-\infty}^{+\infty} \overline{|P(s)|^2}\, ds}. \tag{16}$$

The system averages $\overline{\rho_E(\xi)}$ and $\overline{|P(s)|^2}$ will be smooth curves which follow the general trend of the curves of figure 1, but those irregularities which represent variations from one sample to the next will be smoothed out.

In this way we relate the average angular power spectrum $\overline{|P(s)|^2}$ to the average auto-correlation function $\overline{\rho_E(\xi)}$ for a series of random distributions of electric-wave field.

(*c*) *An optical example*

An example of the type of phenomenon considered in the preceding subsection, drawn from the field of optics, is available in Young's method for the determination of the diameters of small dust particles of approximately the same size (Wood 1934). In this method the dust is sprinkled over a sheet of glass and the Fraunhofer diffraction pattern is measured. The diffracting screen then consists of a series of similar obstructions randomly distributed, and it can be shown that the disturbance over the diffracting screen due to all the obstructions has the same auto-correlation function as that due to one of the obstructions. From our theorem that the angular spectrum depends on the auto-correlation function over the wave-front as it leaves the diffracting screen we therefore deduce that the distribution of power in the angular spectrum is the same as we should have in the angular spectrum of one of the obstructions by itself. A similar argument of course applies to the halo produced by droplets situated at random positions in a cloud (Wood 1934), and to the X-ray diffraction pattern produced by a set of molecules distributed at random in a gas.

(*d*) *Degree of roughness of a diffracting screen*

Suppose that the ground-glass screens considered in subsection (*a*) have irregularities of all sizes and shapes, but not showing much detail of size less than about $a\lambda$. Then the auto-correlation function across the wave emerging from the screens might, when averaged over all the screens, be Gaussian as given by

$$\overline{\rho_E(\xi)} = \exp(-\xi^2/2a^2),$$

and if that were so the angular distribution of power $\overline{|P(s)|^2}$ averaged over all screens would be proportional to the Fourier transform of $\overline{\rho_E(\xi)}$, and we should have

$$\overline{|P(s)|^2} \propto \exp(-2\pi^2 a^2 s^2). \tag{17}$$

† This expression is true because $\int_{-\infty}^{+\infty} |P(s)|^2\, ds$ has the same value for all systems.

The smaller the size of the individual roughnesses, the wider is the corresponding spread of illumination in the angular spectrum, and when the size of the roughness is of the order of $\frac{\lambda}{2\pi}\left(a=\frac{1}{2\pi}\right)$ the spectrum spreads to angles such that $s=\pm 1$ and fills the whole region to $\pm\frac{1}{2}\pi$ on the far side of the screen. We see that the angle over which the (averaged) spectrum spreads depends on the fineness of the grain in the diffracting screen. If the grain is so fine that there is no correlation between points separated by a distance much less than a wave-length then $a \ll 1$ and the angular power spectrum $\overline{|P(s)|^2}$ averaged over systems is seen from equation (17) to be practically constant over the range of real angles given by $-1 < s < +1$. The screen therefore 'scatters' light in such a way that the power $\overline{|P(s)|^2}\,d\theta$ received in any direction is simply proportional to $d\theta$. We shall call screens of this type 'completely rough' and shall return to them in § 8.

(e) *The angular correlation in the spectrum*

Having discussed what determines the shape of the curves of $\overline{\rho_E(\xi)}$ and $\overline{|P(s)|^2}$ averaged over systems we now return to consider the irregular curves of figure 1 which represent $\rho_E(\xi)$ and $|P(s)|^2$ for one particular screen. We first ask the question 'How quickly does the curve for $|P(s)|^2$ fluctuate as s is varied?' or 'If we observe a given value of $|P(s)|^2$ in a given direction s, by how much must we increase s on the average to observe an unrelated value of $|P(s)|^2$?'

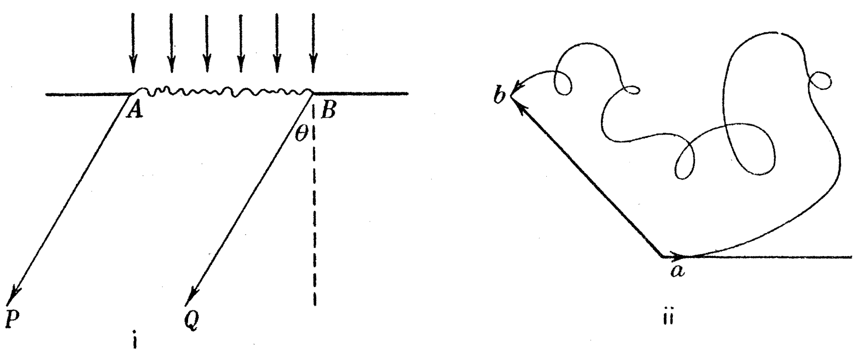

FIGURE 2. To illustrate correlation in the angular spectrum from an irregular screen limited by a stop. (i) illustrates the light incident normally and diffracted at an angle θ from an irregular screen AB. (ii) is an amplitude-phase diagram for the direction AP in (i).

The following simple physical argument will lead us to the correct answer to these questions. The disturbance $P(s)$ corresponding to the direction s $(=\sin\theta)$ in the angular spectrum may be represented as the closing side of an (irregular) amplitude-phase diagram drawn, in the usual way, to represent the diffraction from the screen in the direction s, as shown in figure 2. In this figure the phase of the disturbance from A is taken as a reference and is drawn at a along the direction of zero phase, and the phase of the disturbance from B is represented by the direction of the last element at b on the amplitude-phase diagram. Now let the direction of observation, θ, be increased slightly so that all the elementary vectors in the amplitude-phase diagram alter their directions, the alteration being greatest at the end b of the diagram. It is then reasonable to suppose that the magnitude of the resultant ab cannot change very much until the element at b has rotated through an angle comparable with π. We deduce that, in order to experience an important change in $P(s)$ we must alter s by an amount sufficient to introduce, into the path difference between the ends of the slit, an extra length of about half a wave-length. But this is also the change in s required to take us from a maximum to a minimum in the diffraction pattern produced by the optical stop itself when it is illuminated uniformly. We therefore conclude that when s is varied the change in $P(s)$ is related to the angular diffraction pattern of the optical stop which bounds the original diffracting screen. In fact, we do not expect $P(s)$ to change much until we have moved through an angle equal to that which separates a maximum from the neighbouring minimum in the angular spectrum of the stop.

We shall now deduce this same result more rigorously, and it will be convenient, at the same time, to make the discussion more general. Up to now we have assumed that the series of screens, in whose properties we are interested, are all illuminated by a uniform beam of radiation limited by a straight-sided 'stop', so that the 'system average' of the intensity over the screens, given by $\overline{|E_x(x,0)|^2}$, is constant within the region of the stop and zero outside. We now consider the general case where $\overline{|E_x(x,0)|^2}$ is a known function of any form. Since the averaging over systems has removed the random nature of the screens the remaining variation, represented by $\overline{|E_x(x,0)|^2}$, is due either to a property common to all the screens, or to a variation of the incident radiation across the screens.

Let the field distribution over one of the diffracting screens be given by $E(x)$ and let the angular spectrum be given by $P(s)$, and define the auto-correlation function $\rho_P(\sigma)$ within this angular spectrum by

$$\rho_P(\sigma) = \frac{\int_{-\infty}^{+\infty} P^*(s)\, P(s+\sigma)\, ds}{\int_{-\infty}^{+\infty} |P(s)|^2\, ds}; \tag{18}$$

$\rho_P(\sigma)$ will then be an inverse measure of the average amount by which $P(s)$ changes when we make a change σ in the value of s and will therefore give us what we want, the 'scale' of the irregularities in the curve of $P(s)$. Now we know that $E(x)$ and $\frac{1}{\lambda}P(s)$ are Fourier transforms of each other (see equation (5)), and so we can use equations (11) and (12) to rewrite (18) thus:

$$\rho_P(\sigma) = \frac{\int_{-\infty}^{+\infty} |E(x)|^2 \exp(-2\pi i\sigma x)\, dx}{\int_{-\infty}^{+\infty} |E(x)|^2\, dx}. \tag{19}$$

We now average over systems and since $\int_{-\infty}^{+\infty} |E(x)|^2\, dx$ is the same for all systems, we obtain

$$\overline{\rho_P(\sigma)} = \frac{\int_{-\infty}^{+\infty} \overline{|E(x)|^2} \exp(-2\pi i\sigma x)\, dx}{\int_{-\infty}^{+\infty} \overline{|E(x)|^2}\, dx}. \tag{20}$$

Now, $\overline{|E(x)|^2}$ represents the distribution of illumination over the diffracting screen, averaged over systems; in the case we have previously considered, where the screen was delimited by a stop, $\overline{|E(x)|^2}$ is uniform inside the stop and zero outside, in other cases $\overline{|E(x)|^2}$ might be a function representing the fact that all systems were illuminated by a non-uniform beam of parallel light. Equation (20) then tells us that *the average auto-correlation function* $\overline{\rho_P(\sigma)}$ *within the angular spectrum is proportional to the Fourier transform of the average distribution* $\overline{|E(x)|^2}$ *over the diffracting screens*.

But the Fourier transform of $\overline{|E(x)|^2}$ also represents the angular spectrum produced by a distribution of field which has this same form, and we therefore deduce that the auto-correlation function within the angular spectrum has the same form as the angular spectrum which would be produced by the distribution of illumination given by $\overline{|E(x)|^2}$. This statement clearly agrees with the result of our previous physical argument applied to the case of a simple 'stop'.

(*f*) *The resolving power of the observing instrument*

We now return to the question of the 'average' shape of the irregular angular spectrum illustrated in figure 1 and for simplicity we again consider only the case of a simple 'stop'. We have previously considered an 'average' obtained by the process of evaluating the detailed curves for all the separate systems and 'averaging over systems'. It is now apparent that the 'average' curve can be obtained another way. Let us measure the angular distribution of power in the spectrum by means of a measuring device which has a finite angular resolving power so that it accepts the radiation falling inside a narrow range of angles Δs. Then as the device is moved through the angular spectrum it will measure a 'running mean' over the range Δs.

If the irregularities in the angular spectrum are so widely spaced that they are separated by an angle much greater than Δs, as indicated in figure 1, then the 'running mean' over Δs will, to some extent, repeat the irregularities of $P(s)$ without much smoothing. These conditions will hold when there is some considerable degree of correlation in the angular spectrum so that $\rho_P(\Delta s)$ is not much different from unity. The fact that there is considerable correlation in the angular spectrum implies that the stop is not too large.

Next consider what happens as the optical 'stop' is made wider. The irregularities in $P(s)$ will assume a finer structure, corresponding to the increasing fineness of the diffraction pattern of the 'stop' itself. Finally, when the 'stop' is very wide the structure will become so fine that there are many irregularities inside the acceptance angles Δs of any measuring device we care to use, and all that we can determine is the 'running mean' value. In the limit, for an unbounded irregular screen, although there is zero correlation from point to point in the angular spectrum, all that we can observe with an instrument of finite resolving power is the running-mean value and we measure a completely smooth curve.

(*g*) *Summary*

We now summarize our conclusions as follows. We assume that a large number of sample irregular screens, each limited by an optical 'stop' of the same size, are examined. Then

(i) the angular power spectrum $\overline{|P(s)|^2}$ averaged over systems is the Fourier transform of the auto-correlation function $\overline{\rho_E(\xi)}$ of the disturbance at the screen averaged over systems, and its width depends on the 'fineness of grain' of the screens;

(ii) the auto-correlation function $\overline{\rho_P(\sigma)}$ of the spectrum $P(s)$, averaged over samples, is determined by the width of the aperture, and is the same as the angular spectrum of the aperture itself;

(iii) if the angular power-spectrum of any one screen is observed with a device which accepts a range of angles Δs such that $\overline{\rho_P(\Delta s)}$ is small, then the 'running average' recorded by this device will provide a smooth curve which is the same as the curve representing the average $\overline{|P(s)|^2}$ over systems;

(iv) for a screen of infinite extent any measuring system will delineate a smooth angular spectrum, and as the width of the screen is diminished by a stop which is gradually narrowed, irregularities will begin to appear when the angular spectrum of the stop itself has a structure comparable with the acceptance angle of the measuring device.

6. Fresnel diffraction from a known distribution of field

Up to this point we have considered only the angular spectrum, which corresponds in the optical case to the Fraunhofer diffraction pattern. We now consider the Fresnel diffraction pattern which is formed on a plane placed at a finite distance from the diffracting screen. We shall continue to assume that the screen is illuminated with parallel light.

We have shown in equations (5), (6) and (7) that if the distribution of electric field over the screen at $z = 0$ is given by $E_x(x, 0)$, then the field $E_x(x, z)$ anywhere else is given by

$$E_x(x, z) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} P(s)\exp\{2\pi i(sx+cz)\}\,ds,$$

where distances are measured in terms of the wave-length λ. In considering the Fresnel diffraction pattern we are interested in the field $E_x(x, z_0)$ at a plane distant z_0 from the diffracting screen and, since $c = \surd(1-s^2)$, we may write

$$E_x(x, z_0) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} P(s)\exp\{2\pi i z_0\surd(1-s^2)\}\exp(2\pi i s x)\,ds.$$

We will also write

$$Q(s) = P(s)\exp\{2\pi i z_0\surd(1-s^2)\}, \tag{21}$$

and we then have

$$E_x(x, z_0) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} Q(s)\exp(2\pi i s x)\,ds. \tag{22}$$

Equation (22) describes the Fresnel diffraction pattern produced on a plane at a distance z_0 from the diffracting screen. Before we consider it in detail let us investigate more fully the relation between $Q(s)$ and $P(s)$. Equation (21) shows that

$$\left.\begin{aligned} |Q(s)| &= |P(s)| \quad \text{for} \quad |s|<1,\\ \text{and}\qquad |Q(s)| &\ll |P(s)| \quad \text{for} \quad |s|>1,\end{aligned}\right\} \tag{23}$$

provided that z_0 is greater than a few wave-lengths.

Now values of $|s|>1$ correspond, in the spatial Fourier analysis of the field distribution over the diffracting screen, to components which have repetition distances less than the wave-length λ of the radiation. It follows that $|Q(s)|$ is derived from $|P(s)|$ by taking $|Q(s)| = |P(s)|$ for all components in the spatial analysis of the screen which have repetition distances greater than the wave-length λ of the radiation and putting $|Q(s)|$ equal to zero for all other values of s. If, in the Fourier analysis $P(s)$ of the screen, there are no components with $|s|>1$, i.e. with repetition distances less than λ, we may write $|Q(s)| = |P(s)|$. This is the state of affairs with ordinary optical diffraction problems where there is no structure in the diffracting screen comparable in size with the wave-length of the light.

We shall now use expression (22), which gives the field distribution on the plane z_0, to deduce some interesing points about the auto-correlation function $\rho_{Ez_0}(\xi)$ of the Fresnel diffraction pattern $E(x, z_0)$. Equation (22) shows that $\frac{1}{\lambda} Q(s)$ is the Fourier transform of $E(x, z_0)$ and therefore, if $\rho_{Ez_0}(\xi)$ is defined from equation (8) as

$$\rho_{Ez_0}(\xi) = \frac{\int_{-\infty}^{+\infty} E^*(x, z_0)\, E\{(x+\xi), z_0\}\, dx}{\int_{-\infty}^{+\infty} |\, E\{x, z_0\}\,|^2\, dx},$$

equation (13) allows us to write

$$\rho_{Ez_0}(\xi) = \frac{\int_{-\infty}^{+\infty} |\, Q(s)\,|^2 \exp(2\pi i s\xi)\, ds}{\int_{-\infty}^{+\infty} |\, Q(s)\,|^2\, ds}. \tag{24}$$

Let us now write $\rho_{E_0}(\xi)$ to represent the auto-correlation function of the field distribution $E(x, 0)$ at the diffracting screen $z = 0$. Then

$$\rho_{E_0}(\xi) = \frac{\int_{-\infty}^{+\infty} E^*(x, 0)\, E\{(x+\xi), 0\}\, dx}{\int_{-\infty}^{+\infty} |\, E(x, 0)\,|^2\, dx}$$

$$= \frac{\int_{-\infty}^{+\infty} |\, P(s)\,|^2 \exp(2\pi i \xi s)\, ds}{\int_{-\infty}^{+\infty} |\, P(s)\,|^2\, ds}. \tag{25}$$

Since $|\, P(s)\,|$ and $|\, Q(s)\,|$ are closely related equations (24) and (25) show that there is a close relation between the auto-correlation function $\rho_{Ez_0}(\xi)$ of the Fresnel diffraction pattern and that $\rho_{E_0}(\xi)$ of the field distribution over the diffracting screen. In the optical case we have already seen that $|\, P(s)\,| = |\, Q(s)\,|$, and it follows that $\rho_{Ez_0}(\xi) = \rho_{E_0}(\xi)$, or, in words, that *the generalized auto-correlation function of the Fresnel diffraction pattern is the same as that of the field distribution which gives rise to it.*

The argument of this section can be summarized simply as follows. The field distribution over all space, including the diffracting screen, is known when the angular spectrum is given, and this angular spectrum is the same everywhere (except quite close to the diffracting screen, where it may contain some 'evanescent' waves if the screen has some structure of size less than the wave-length). But the auto-correlation function over any plane parallel to the screen depends only on the angular spectrum and hence it is the same over all such planes (Fresnel diffraction patterns) provided they are not too near the screen, and it is the same as that of the screen itself when the latter has no structure of size less than one wave-length.

This conclusion, appropriate to the optical case, has been tested as follows. The Fresnel diffraction pattern of a slit was computed from tables of Fresnel integrals in the ordinary way, so that both the amplitude and phase at each point of the diffraction pattern was known. The generalized auto-correlation function as defined in equation (8) was then computed, and was compared with that of the slit itself. In the case considered the slit was taken to be 20 wave-lengths wide and the Fresnel diffraction patterns and their auto-correlation functions were computed at distances of 50 and 200 wave-lengths. The results are shown in figure 3. The values of $\rho_E(\xi)$, the auto-correlation function, are seen to be nearly the same for $z = 0$, $z = 50$ and $z = 200$. For the following reasons we do not expect the agreement to be exact:

(i) We have used for our calculation only the central portion of the diffraction pattern, as shown in figure 3.

(ii) Detail of size less than a wave-length in the aperture distribution is not reproduced in the Fresnel diffraction pattern; for example, the sharp edges of the function shown in figure 3*a* would be of this nature.

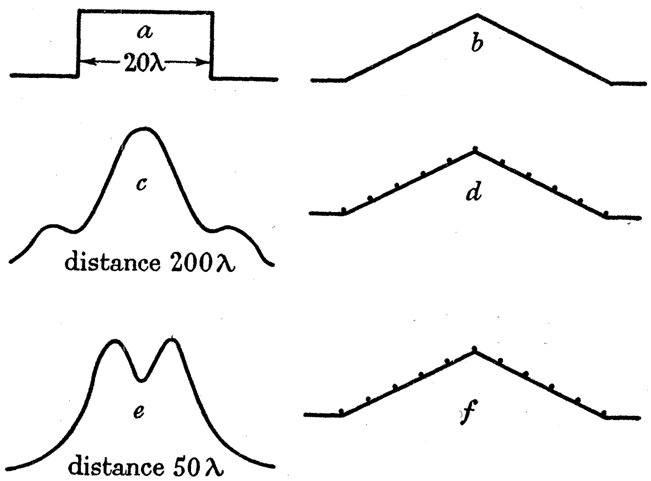

FIGURE 3. Generalized auto-correlation functions of Fresnel diffraction patterns produced by a slit. *a* represents the distribution of amplitude over a slit 20λ wide; *b* represents the auto-correlation function of this distribution; *c* and *e* represent the distributions of amplitude in the Fresnel diffraction patterns at distances of 200 and 50λ; in *d* and *f* the auto-correlation functions calculated for the diffraction patterns of *c* and *e* respectively are shown as dots. The continuous lines repeat the auto-correlation function shown at *b*.

7. Fresnel diffraction from a random distribution of field

In the radio problem with which we are concerned it is ideally possible to observe the field produced at all points of the earth's surface when a wave originating from a small aerial (a point source) is reflected from the ionosphere. A close approximation to this problem is one in which a parallel beam of radiation is incident on the ionosphere from above so as to form a Fresnel diffraction pattern at the ground. If, with this model, we knew the amplitude and phase of the disturbance at all points on the ground we could evaluate its generalized (complex) correlation function and, according to the theorem of the preceding section, this would give us information about the auto-correlation function of the wave as it emerged from the ionosphere.

If the field distribution over a plane below the ionosphere is random in the sense discussed in § 5 (*a*), we find that a valuable theorem can be stated which relates the *amplitude* in the Fresnel diffraction pattern to the statistical properties of the random field distributions. This is in contrast to the theorem of § 6 which relates the generalized auto-correlation function (which is derived from a knowledge of both the *amplitude* and the *phase*) of the Fresnel diffraction pattern to the completely determined field distribution which gives rise to it. We now proceed to develop the relation appropriate to a random series of field distributions.

The received e.m.f. $E_x(x, z_0)$ at a point x on the ground at a distance z_0 from a diffracting ionosphere has been shown (equation (22)) to be given by

$$E_x(x, z_0) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} Q(s)\exp\{2\pi isx\}\,ds.$$

If we introduce the factor $\exp\{-2\pi if_0 t\}$ we can write the electric field $\mathscr{E}_x(x, z_0, t)$ as a function of space and time as follows:

$$\mathscr{E}_x(x, z_0, t) = \frac{1}{\lambda}\mathscr{R}\left[\int_{-\infty}^{+\infty} Q(s)\exp(2\pi isx - 2\pi if_0 t)\,ds\right]. \tag{26}$$

A stationary observer at the point (x, z_0) will observe $\mathscr{E}_x(x, z_0, t)$ as a sinusoidal oscillation of fixed amplitude R and phase θ, where $R = |E_x(x, z_0)|$ and $\theta = -\arg\{E_x(x, z_0)\}$, and for our purpose it is required to investigate the way in which R varies over the plane z_0 as x is varied. For this purpose we shall find it convenient to consider the field experienced by an observer moving along $z = z_0$ with a velocity v; he will observe a quasi-oscillatory e.m.f. whose amplitude varies in a random manner in time, and the time variation will be related to the space variation which we require. This method of approach is convenient because it will lead to the derivation of a result which we require later when considering the problem of fading (§ 10).

Suppose, then, that an observer moves with velocity v along $z = z_0$. The field $\mathscr{E}'_x(z_0, t)$ which he experiences can be deduced from equation (26) by writing $\lambda x = vt$ so that

$$\mathscr{E}'_x(z_0, t) = \frac{1}{\lambda}\mathscr{R}\left[\int_{-\infty}^{+\infty} Q(s)\exp\left\{2\pi it\left(-f_0 + \frac{sv}{\lambda}\right)\right\}ds\right].$$

Let us now put $-f_0 + sv/\lambda = f$, to obtain

$$\mathscr{E}'_x(z_0, t) = \frac{1}{v}\mathscr{R}\left[\int_{-\infty}^{\infty} Q\left(\frac{\lambda(f+f_0)}{v}\right)\exp(2\pi ift)\,df\right]. \tag{27}$$

Now the phase of $Q(s)$ is assumed to be completely random, so that equation (27) could be thought of as describing the e.m.f. due to electrical noise in which the power spectrum is given, as a function of frequency, by $|Q\{\lambda(f+f_0)/v\}|^2$. Noise of this type could be produced, for example, by passing 'white' electrical noise through the appropriate type of filter. In our case the amplitude of $Q(s)$ is zero except when $|s| < 1$, and therefore the power spectrum is narrow compared with f_0 provided that $v \ll \lambda f_0 = c$, the velocity of light.

A detailed discussion of an e.m.f. of the type described by equation (27) is given in the appendix. It is there shown that the e.m.f., which corresponds to 'white' noise after passage through a narrow filter centred on a frequency f_0, is equivalent to an approximately sinusoidal e.m.f. with a slowly varying amplitude $R(t)$, so that we may write

$$\mathscr{E}'_x(z_0, t) = E'(t) = \mathscr{R}[R(t)\exp\{2\pi if_0 t + i\theta(t)\}].$$

It is further shown that the auto-correlation function $\rho_R(\tau)$ of the amplitude R of the fluctuation defined by

$$\rho_R(\tau) = \frac{\int_{-\infty}^{+\infty} R(t)\,R(t+\tau)\,dt - \left\{\int_{-\infty}^{+\infty} R(t)\,dt\right\}^2}{\int_{-\infty}^{+\infty} R^2(t)\,dt - \left\{\int_{-\infty}^{+\infty} R(t)\,dt\right\}^2}, \tag{28}$$

is given in terms of the power spectrum by the expression

$$\rho_R(\tau) = \frac{\left\{\left|\int_{-\infty}^{\infty} |\, Q(\lambda f/v)\,|^2 \exp(2\pi i f\tau)\, df\right|\right\}^2}{\left\{\int_{-\infty}^{\infty} |\, Q(\lambda f/v)\,|^2\, df\right\}^2}. \tag{29}$$

We now return from the time-varying amplitude experienced by the moving observer, to the space-varying amplitude which actually exists, and deduce the spatial auto-correlation function $\rho_R(\xi)$ of R by writing $\xi\lambda = \tau v$ and $\lambda f/v = s$ to obtain

$$\rho_R(\xi) = \frac{\left\{\left|\int_{-\infty}^{\infty} |\, Q(s)\,|^2 \exp(2\pi i s\xi)\, ds\right|\right\}^2}{\left\{\int_{-\infty}^{\infty} |\, Q(s)\,|^2\, ds\right\}^2}, \tag{30}$$

where

$$\rho_R(\xi) = \frac{\overline{R(x)\, R(x+\xi)} - \{\overline{R(x)}\}^2}{\overline{R^2(x)} - \{\overline{R(x)}\}^2}. \tag{31}$$

Equation (30) shows that $\rho_R(\xi)$ depends only on $|\, Q(s)\,|$, and since $|\, Q(s)\,|$ is the same over all planes, except near the diffracting screen itself, we deduce that $\rho_R(\xi)$ is the same for all the Fresnel diffraction patterns, provided that they are not taken too close to the diffracting screen. Moreover, in the case where $|\, P(s)\,| = |\, Q(s)\,|$, i.e. when the screen has no structure of size comparable with a wave-length, the function $\rho_R(\xi)$ is the same over the screen as over the Fresnel diffraction pattern. We can state these conclusions as follows.

The auto-correlation function of the *amplitude* over the Fresnel diffraction pattern formed by a random screen is the same over all such patterns provided that observations are only made at distances greater than a few wave-lengths from the screen. The auto-correlation function is also the same as that over the screen itself when either (*a*) the screen has no structure of size comparable with a wave-length, or (*b*) structure of this size is neglected in computing the auto-correlation function.

8. Diffraction by a 'completely rough' screen

It is of interest to enquire what kind of rough surface would give rise to 'Lambert's law' of scattering. For the two-dimensional case which we have considered in this paper we shall assume that Lambert's law is obeyed if, when the screen is illuminated normally, each element of area dS scatters in such a way that the power crossing unit area of a receiver at a distance ρ is proportional to $dS\cos\theta/\rho$, where θ is the angle between the normal and the direction of scattering. We shall, moreover, consider only the case where the wave is polarized with its electric vector along the X-direction, that is, the direction along which the wave-field is assumed to be changing.

For the two-dimensional case considered the quantity $dS\cos\theta/\rho$ is equal to $d\theta$, the angle subtended at the receiver by the element dS, so that Lambert's law implies that the power received is proportional to the range of angles $d\theta$ to which the receiver is sensitive or that the power received per unit range of angles in the angular spectrum is constant. Now, the argument of § 3 shows that this implies that $|\, F_1(\theta)\,|$ is constant and therefore, from equation (4) (i), that $|\, G_x(s/\lambda)\,| = |\, P(s)\,|$ is constant. But if $|\, P(s)\,|$ is constant equation (15) shows that $\rho_E(\xi)$ is a narrow function and that there is little correlation between the field E_x at two points however close together, and the screen producing the field might be said to be 'completely rough'. It therefore appears that a screen which is 'completely rough' in this sense will give Lambert's law of scattering.

It is also of interest to consider the Fresnel diffraction pattern which would be produced by this screen. Since $|P(s)|$ is constant, we have from equation (23)

$$|Q(s)| = \text{constant} \quad \text{for} \quad |s|<1,$$
$$|Q(s)| \doteqdot 0 \quad \text{for} \quad |s|>1,$$

and hence, from equation (30),

$$\rho_R(\xi) \propto \left[\int_{-1}^{+1} \exp(2\pi i s\xi)\, ds\right]^2 \propto \left[\frac{\sin(2\pi\xi)}{2\pi\xi}\right]^2. \tag{32}$$

The function (32) should therefore represent the auto-correlation function of the Fresnel diffraction pattern produced by an ideally 'rough' screen of the type here discussed. This function falls to zero when $\xi = \frac{1}{2}$, i.e. when the separation is equal to half the wave-length. For a screen of this kind we should therefore expect that the Fresnel diffraction pattern would show small correlation of amplitudes at points separated by more than $\frac{1}{2}\lambda$. On some occasions it appears that the ionosphere behaves approximately like a 'completely rough' diffractive reflector of this type, for waves of length greater than about 200 m.

PART II. THE FADING PRODUCED BY A CHANGING IONOSPHERE

9. Outline of the problem

If the irregular ionosphere, or the irregular diffracting screen, is not constant, but varies with time, the resultant wave-fields at the ground will also vary, and we now proceed to examine the variations which would be produced by different ionospheric movements.

Two extreme types of movement are discussed; in one the irregularities are assumed to move with random velocities with mean velocity zero, and in the other the irregular ionosphere is assumed to move overhead with a constant velocity, but so as to maintain its original form. Combinations of the two types of movement are not dealt with here, but are discussed in a paper by Briggs, Phillips & Shinn (1950).

The calculation in the case of the irregular movement follows that of Ratcliffe (1948), who evaluated the Doppler changes of frequency due to the differently moving ionospheric irregularities and deduced the 'power spectrum' of the returned signal. This power spectrum is then related to the auto-correlation function of the fading curve by a theorem proved in the appendix. The calculation in the case of a steady drift of the ionosphere proceeds in the same way.

An alternative approach to the case of a steadily drifting ionosphere is also discussed in § 12. In this we assume a knowledge of the diffraction pattern produced on the ground by a steady ionosphere, and consider what would happen if this was caused to drift at uniform speed past an observer. It is shown that the two methods of approach lead to the same result.

In § 13 similar methods of calculation are applied to problems of the fading of an echo which has been reflected in succession from the ionosphere, the ground, and the ionosphere a second time; and in § 14 an outline is given of the problem of the fading of a wave incident obliquely on the ionosphere.

10. General method of calculation

In what follows we shall make in succession a series of different assumptions about the movements of portions of the ionosphere, and shall evaluate the Doppler shift of frequency produced in a wave scattered by the portion concerned. We shall deduce, from the details of the model we are discussing, the power scattered at each frequency (f), and we shall assume that the different parts of the ionosphere scatter incoherently, so that the powers add. We shall then be in a position to write down an expression for the 'power spectrum' $W(f)$ of the wave returned from the ionosphere.

In the models which we shall discuss the changes of frequency produced by the Doppler effects will be small compared with the radio-frequencies in question, and since we shall assume a statistical equality between the numbers of approaching and receding scattering centres the power spectrum will also be symmetrical about the original radio-frequency f_0. The power spectrum $W(f)$ will therefore be a narrow function centred on the original radio-frequency f_0, and the phases of the component e.m.f.'s will be random. We have already dealt with an e.m.f. of this type (§ 7) and have noticed that it is similar to the 'radio noise' e.m.f. which would result if a 'white' source of radio noise were connected to a narrow-band filter which had a 'power spectrum' given by $W(f)$. The resulting e.m.f. $E(t)$ will be a quasi-sinusoidal oscillation of frequency f_0 whose amplitude $R(t)$ and phase $\theta(t)$ vary slowly with time, at rates small compared with the rate of change of the original radio frequency e.m.f. $\cos 2\pi f_0 t$. It will be given by

$$E(t) = \mathscr{R}[R(t) \exp\{i(2\pi f_0 t+\theta(t))\}]. \tag{33}$$

This type of e.m.f. is considered in the appendix, and it is shown that the auto-correlation function $\rho_R(\tau)$ of R defined by

$$\rho_R(\tau) = \frac{\int_{-\infty}^{+\infty} R(t)\, R(t+\tau)\, dt - \left\{\int_{-\infty}^{+\infty} R(t)\, dt\right\}^2}{\int_{-\infty}^{+\infty} \{R(t)\}^2\, dt - \left\{\int_{-\infty}^{+\infty} R(t)\, dt\right\}^2}, \tag{34}$$

is given in terms of the power spectrum $W(f)$ by equation (53) which we will re-write as

$$\rho_R(\tau) = \frac{\left\{\left|\int_{-\infty}^{+\infty} W(f_0+f) \exp(2\pi i f\tau)\, df\right|\right\}^2}{\left\{\int_{-\infty}^{+\infty} W(f_0+f)\, df\right\}^2}. \tag{35}$$

We also note that, if Wiener's theorem (our equation (14)) is used, we can deduce the auto-correlation function $\rho_E(\tau)$ of the instantaneous e.m.f. $E(t)$ from the power spectrum. Defining $\rho_E(\tau)$ by the expression

$$\rho_E(\tau) = \frac{\int_{-\infty}^{+\infty} E(t)\, E(t+\tau)\, dt}{\int_{-\infty}^{+\infty} \{E(t)\}^2\, dt}, \tag{36}$$

we find from equation (14) that

$$\rho_E(\tau) = \frac{\int_0^\infty W(f)\cos(2\pi f\tau)\,df}{\int_0^\infty W(f)\,df}. \tag{37}$$

In the rest of this paper we shall make considerable use of (35), and it may be useful to illustrate its meaning by reference to figure 4 in which *a* represents the e.m.f. $E(t)$ and the amplitude $R(t)$ of 'noise' which has passed through a narrow filter passing frequencies

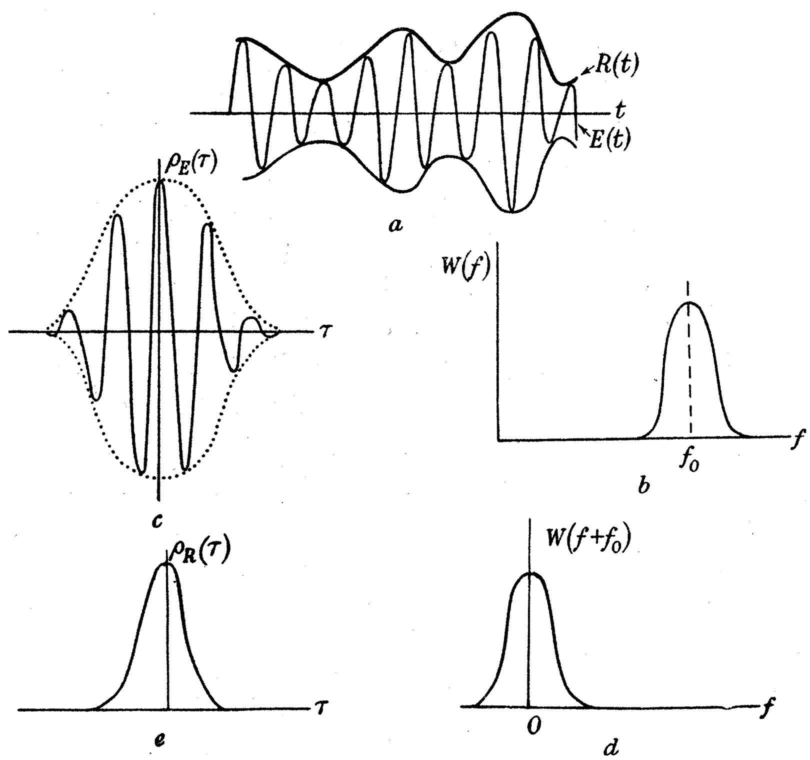

FIGURE 4. To illustrate the analysis of a quasi-periodic e.m.f. with 'random' amplitude. *a*, time variation of e.m.f. $E(t)$ and amplitude $R(t)$; *b*, power spectrum of $E(t)$; *c*, the solid line represents the auto-correlation function $\rho_E(\tau)$ of $E(t)$ and is the Fourier transform of the curve in *b*; *d*, power spectrum $W(f+f_0)$ of $E(t)$ after it has been shifted by f_0 (compare *b*); *e*, the auto-correlation function $\rho_R(\tau)$ of $R(t)$. This function is the square of the function shown dotted in *c* and is the square of the Fourier transform of the function shown in *d*.

near f_0, as shown by the spectrum of power $W(f)$ in (*b*). Figure 4*c* shows the auto-correlation function $\rho_E(\tau)$ of the e.m.f. $E(t)$, and, according to equation (37), $\rho_E(\tau)$ is proportional to the Fourier transform of $W(f)$. We, of course, expect $\rho_E(\tau)$ to be quasi-periodic, since $E(t)$ is itself quasi-periodic. In figure 4*d* we represent the function $W(f+f_0)$, and in figure 4*e* the function $\rho_R(\tau)$ which, by equation (35), is proportional to the square of the Fourier transform of $W(f+f_0)$. Since $R(t)$ is not in any way periodic we do not expect $\rho_R(\tau)$ to be periodic. We note also that the function $\rho_R(\tau)$, depicted in *e*, is the square of the envelope of the function $\rho_E(\tau)$ shown dotted in *c*.

In what precedes we have assumed that the power spectrum $W(f)$ is symmetrical about f_0, and this assumption is justified in most circuits and in the cases which we shall consider here. It is interesting to note that if there is no frequency about which $W(f)$ is symmetrical it is still possible to derive a useful expression for $\rho_R(\tau)$ in the form

$$\rho_R(\tau) \propto \left| \int_{-\infty}^{\infty} W(f+f_0) \exp(2\pi i f\tau)\, df \right|^2,$$

where f_0 is any arbitrary frequency, including zero. We now proceed to use equation (35) to make deductions about the nature of the fading produced by some theoretical models of the ionosphere.

11. Scattering from irregularities which have random velocities

We now consider the case of a model irregular ionosphere on which a radio wave of frequency f_0 is incident and first suppose that the wave is returned by scattering from a series of irregularities which have velocities distributed like those of the molecules in a gas. The velocities v along a line of sight are then distributed with a probability $p(v)$ and with a r.m.s. velocity v_0 as given by

$$p(v) \propto \exp\left(-\frac{v^2}{2v_0^2}\right). \tag{38}$$

The wave of frequency f_0 returns with frequency $f = f_0 \pm 2v/\lambda$ after reflexion from one of the irregularities, and substitution in equation (38) shows that the probability $p_1(f)$ that a frequency f is returned is given by

$$p_1(f) = \exp\left\{-(f-f_0)^2 \frac{\lambda^2}{8v_0^2}\right\}. \tag{39}$$

On the assumption that each irregularity scatters the same power incoherently this expression also gives the power spectrum of the returned wave. From equation (35) we therefore deduce

$$\rho_R(\tau) \propto \left\{\int_{-\infty}^{+\infty} \exp\left(-\frac{f^2\lambda^2}{8v_0^2}\right) \exp(2\pi i f\tau)\, df\right\}^2.$$

Therefore

$$\rho_R(\tau) \propto \left\{\exp\left(-\frac{\pi^2\tau^2 8v_0^2}{\lambda^2}\right)\right\}^2$$

$$\propto \exp\left(-\frac{16\pi^2 v_0^2 \tau^2}{\lambda^2}\right). \tag{40}$$

Equation (40) gives a description of the irregular time fluctuation (fading) of the wave returned from an ionosphere of this kind. The application of this expression to experimental results is discussed shortly in § 15.

It is of interest to consider the 'rate of fading' of the irregular type of signal variation which is produced by the ionospheric model here discussed. Equation (40) shows that the auto-correlation function $\rho_R(\tau)$ falls to e^{-1} after a time $\tau = \lambda/4\pi v_0$, and we might call this the 'fading time', since it will on the average represent the time which must elapse before the received signal has altered appreciably. We note that if there were only two scattering centres, one stationary and one moving with velocity v_0 in the line of sight, the fading would be periodic with period $\lambda/2v_0$.

The fading appropriate to other more complicated models can be deduced in a manner similar to the above. Thus if the individual irregularities do not all scatter the same power, but are so arranged that for each velocity v there is a large number of irregularities with a random distribution of scattering power, equation (39) still holds and the conclusion of equation (40) is valid.

If the power scattered from the greater distances is less than that scattered from the nearer, and if at each distance there is a large number of scattering irregularities with the same distribution of velocities in the line of sight, the expressions are again valid.

12. Scattering from an irregular ionosphere moving with a steady velocity

We next investigate the fading which is produced by an irregular ionosphere which keeps a constant form but moves with a velocity v. The problem can be reduced to a simple one in which a one-dimensional irregular screen is illuminated by a parallel beam of radiation, and we wish to evaluate the nature of the 'fading' observed at a point below the screen as it moves with uniform velocity v parallel to its length.

For this purpose we first analyze the field, in the region below the diffracting screen, into an 'angular spectrum' $P(\sin\theta)$ of plane waves travelling in different directions θ, as indicated in § 3.

A translational movement of the screen is equivalent to a movement of the observer, and each component plane wave in the angular spectrum undergoes a Doppler shift of frequency so that its frequency f becomes

$$f = f_0 + \frac{v\sin\theta}{\lambda} \quad \{-1 < \sin\theta < +1\}.$$

The frequency spectrum of the power $W(f)$ in the diffracted wave can therefore be written

$$W(f) = \left| P\left(\frac{\lambda}{v}(f-f_0)\right)\right|^2. \tag{41}$$

It is convenient to discuss our problem in terms of an ionosphere which is 'completely rough' in the sense defined in § 8, though the method of calculation is general and is not limited to that case. For a 'completely rough' ionosphere we have seen that

$$|P(\sin\theta)|^2 = \begin{cases} \text{constant for } -1 < \sin\theta < +1 \\ 0 \text{ for } \sin\theta \text{ outside this range} \end{cases}.$$

Equation (41) then shows that $W(f)$ is constant for frequencies in the range

$$\left(f_0 - \frac{v}{\lambda}\right) < f < \left(f_0 + \frac{v}{\lambda}\right).$$

The function $W(f)$ is shown in figure 5*a*.

Now from the power spectrum of figure 5*a* we use the results of equation (35) to deduce the auto-correlation function $\rho_R(\tau)$ of the amplitude of the resulting fading curve, and arrive at the expression

$$\rho_R(\tau) \propto [\text{Fourier transform of } W(f+f_0)]^2$$
$$\propto \left\{\int_{-v/\lambda}^{+v/\lambda} \exp(2\pi i f\tau)\, df\right\}^2,$$

or,

$$\rho_R(\tau) \propto \frac{\sin^2(2\pi v\tau/\lambda)}{(2\pi v\tau/\lambda)^2}. \tag{42}$$

The functions $W(f)$, $W(f+f_0)$ and $\rho_R(\tau)$ are shown in figure 5.

It is interesting to derive equation (42) in a different way. If the irregular screen were stationary it would produce on the ground an irregular diffraction pattern, and it was shown in § 8 that the spatial auto-correlation function of this pattern is given by

$$\rho_R(\xi) = \frac{\sin^2 2\pi\xi}{(2\pi\xi)^2}. \tag{43}$$

If now the screen, and with it the diffraction pattern, moves with a velocity v, an observer will experience fading as the diffraction pattern moves past him, and the auto-correlation function of the fading in time is related to the spatial correlation function by writing $\xi\lambda = v\tau$. Equation (43) is then seen to be equivalent to equation (42).

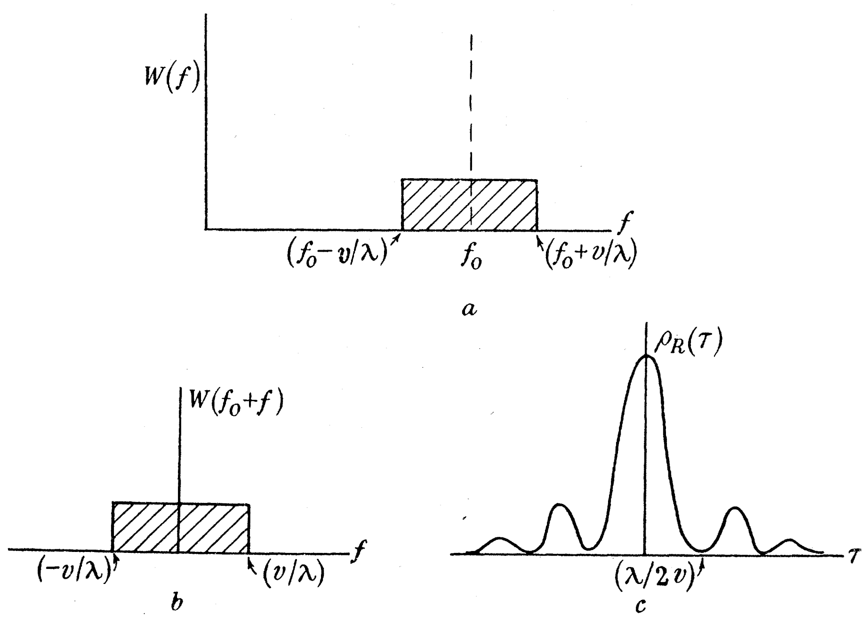

FIGURE 5. The fading produced by a 'completely rough' ionosphere moving with horizontal velocity v. a, the power spectrum of the reflected wave; b, the power spectrum shifted to be centred on zero; c, $\rho_R(\tau)$ which is the square of the Fourier transform of b.

13. THE FADING OF A DOUBLY REFLECTED WAVE

It is of interest to consider the fading of a wave which has been reflected in succession from the ionosphere, the ground, the ionosphere a second time and back to the ground. The wave is emitted with a frequency f_0, and on its first return to the ground it has a power spectrum $W(f)$ which is spread in a narrow range round f_0 so that it can be represented by a function $V(f-f_0)$, where

$$V(f-f_0) = W(f) \quad \text{or} \quad V(f) = W(f+f_0).$$

The power $V(f-f_0)\,df$ in each frequency range f to $f+df$ now behaves like a new effective transmitter at the ground, and after the second scattering from the ionosphere it is spread over a range of frequencies f', centred on f, as given by the function $V(f'-f)$. The final power $W_2(f')\,df'$ in the frequency range f' to $f'+df'$, derived from the effective ground transmitters in the frequency range f to $f+df$, is therefore given by

$$W_2(f')\,df' = V(f'-f)\,V(f-f_0)\,df\,df',$$

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so that the total power in the frequency range f' to $f'+df'$, derived from 'effective ground transmitters' of all frequencies, is given by

$$W_2(f')\,df' = \left[\int V(f'-f)\,V(f-f_0)\,df\right]df'. \quad (44)$$

Now write $f-f_0 = \alpha$ and $f'-f_0 = F$ and (44) becomes

$$W_2(f') = \int V(\alpha)\,V(F-\alpha)\,d\alpha. \quad (45)$$

We now use the theory of convolutions to transform the integral as follows:

$$\begin{aligned} W_2(f') &= \int V(\alpha)\,V(F-\alpha)\,d\alpha \\ &= \int \{v(t)\}^2 \exp(-2\pi i F t)\,dt, \end{aligned} \quad (46)$$

where $v(t)$ is the Fourier transform of $V(F)$.

Now for use in equation (35) we require $W_2(f'+f_0)$, and to evaluate this we apply the 'shift' theorem of Fourier analysis to equation (46) and obtain

$$\begin{aligned} W_2(f'+f_0) &= \int \{v(t)\}^2 \exp(-2\pi i F t)\exp(-2\pi i f_0 t)\,dt \\ &= \int \{v(t)\}^2 \exp(-2\pi i f' t)\,dt, \end{aligned}$$

so that $\{v(t)\}^2$ is the Fourier transform of $W_2(f'+f_0)$. Equation (35) can now be used to show that the auto-correlation function $\rho_{R(2)}(\tau)$ for the doubly reflected wave is given by the expression

$$\rho_{R(2)}(\tau) \propto \{|v(\tau)|\}^4. \quad (47)$$

Application of equation (35) to the case of the singly reflected wave shows that its auto-correlation function $\rho_{R(1)}(\tau)$ is given by

$$\begin{aligned} \rho_{R(1)}(\tau) &\propto \{|\text{Fourier transform of } W(f_0+f)|\}^2 \\ &\propto \{|\text{Fourier transform of } V(f)|\}^2 \\ &\propto \{|v(t)|\}^2. \end{aligned} \quad (48)$$

Comparison of (47) and (48) shows us that

$$\rho_{R(2)}(\tau) = \{\rho_{R(1)}(\tau)\}^2,$$

whatever mechanism we assume for the production of the fading. We thus arrive at the conclusion that the auto-correlation function of the amplitude of the fading curve of the doubly reflected echo is the square of the auto-correlation function for the singly reflected echo. This result is independent of the precise movements postulated to explain the fading provided only that these are random in direction. This provision is necessary because it is possible to envisage conditions under which the frequency shift imposed by the second reflexion is not the same as that imposed by the first reflexion. For example, if the ionosphere consisted of irregularities all moving with the same horizontal velocity, the rate of fading of the second reflected signal would be less than that given by the above result.

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14. Oblique transmission

Although it is not, in general, an easy matter to extend the analysis of this paper to cover the case of reflexion from the ionosphere at oblique incidence, it is comparatively simple to estimate the way in which the speed of fading depends on the angle of incidence when the fading is produced by irregular movements of scattering centres in the ionosphere. In figure 6 suppose that T and R represent a transmitter and a receiver and AB represents an ionospheric region which contains randomly moving scattering centres. The only movements of a scattering centre C which produce a change in the length of the path from T to R, and therefore a Doppler shift, are those normal to the ellipsoid drawn through C with T and R as foci. A movement of C with a velocity v normal to this ellipsoid leads to a Doppler shift of frequency given by $\frac{2v}{\lambda}\cos\{\frac{1}{2}(\theta+\phi)\}$, where θ and ϕ are the angles shown in the figure. If we assume that the most important scattering centres lie somewhere near the point of symmetry, then the Doppler shift is approximately proportional to $(2v/\lambda)\cos i$, where i is the angle of incidence for a layer AB which acts like a plane reflector. The Doppler shift does not depart markedly from this value even if C is some distance from the point of symmetry. It should be noted that, in this argument, the plane of figure 6 is not necessarily the vertical plane through T and R.

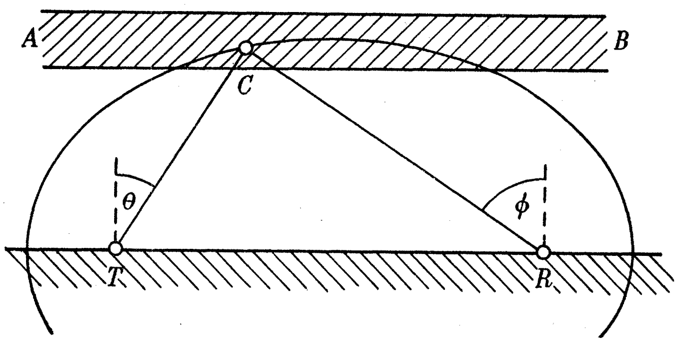

Figure 6. To illustrate oblique reflexion from the ionosphere.

We can now repeat the analysis of § 11 using the effective velocity $v_0 \cos i$ instead of v_0, and we deduce that the auto-correlation function of the temporal variation of signal amplitude is given by

$$\rho_R(\tau) \propto \exp\left\{-\frac{16\pi^2 v_0^2 \tau^2 \cos^2 i}{\lambda^2}\right\}. \tag{49}$$

Equation (49) indicates that the 'fading time', discussed in § 11, is proportional to $(\lambda \sec i)/v_0$. When the distance TR is more than 3 or 4 times the height of the reflecting region, $\sec i$ is approximately proportional to the distance TR of transmission. These considerations appear to provide an explanation for the observed fact that the 'fading time' of radio signals on the same wave-length are known to be roughly proportional to the distance of transmission.

15. Conclusions

It was explained in the introduction that we would not attempt to solve the actual problems occurring in the case of ionospheric diffraction but that we would consider some simpler allied problems in diffraction. With this end in view we have shown (§ 3) how the angular spectrum (polar diagram, Fraunhofer diffraction pattern) of a given field distribution (diffracting screen) can be deduced from the Fourier transform of the distribution of field over the screen. If the angular distribution of power is alone required it is shown (§ 4) that it is sufficient to know the auto-correlation function of the field distribution over the screen.

If, as is often convenient, a random screen is specified in terms of this function we can make deductions about the angular spectrum of *power* from such a screen (§ 5). It is also shown in § 5 that the angular correlation of the spectrum from a random screen depends on the distribution of average field intensity over the screen. When the screen is bounded by a stop but otherwise illuminated uniformly it depends on the size of the stop.

It is shown how the generalized auto-correlation function of a Fresnel diffraction pattern is related to that of the diffracting screen which produces the pattern and the case of optical diffraction from a slit is used as an illustrative example. When the screen is of a random nature it is shown that the auto-correlation function of the *amplitude* in the Fresnel diffraction pattern is related to a similar quantity in the field distribution over the screen. The Fresnel diffraction pattern which would be produced by a completely rough screen, defined in § 8, is comparable with that which is sometimes observed when radio waves are diffractively reflected from the ionosphere and there appear to be some grounds for believing that the ionosphere is occasionally 'completely rough' in this sense.

It has been possible to calculate the kind of irregular fading which would be produced by two types of model, one in which the scattering centres in the ionosphere are moving in random directions with velocities distributed like those of the gas-kinetic motion of gaseous molecules, and the other in whch an ionosphere with a fixed irregular structure moves with a uniform velocity. In each case the temporal auto-correlation function $\rho_R(\tau)$ of the amplitude R is related to the velocity of the ionospheric movements, as shown by equations (40) and (42) for the two cases considered. We have also shown how we should expect the conclusions to be modified if the reflexion from the ionosphere occurs at oblique incidence. In § 13 it has been shown that the auto-correlation function $\rho_{R(2)}(\tau)$ for a wave twice reflected from the ionosphere is equal to the square of the auto-correlation function $\rho_{R(1)}(\tau)$ for a once-reflected wave.

It is not the purpose of this paper to apply these theoretical results to explain observed phenomena, but it should be mentioned that some of our colleagues have applied them to observations they have made, and the results have been discussed in terms of theories of the ionosphere. Thus Ratcliffe (1948) has compared the fading observed when different frequencies are reflected at vertical incidence. McNicol (1949) has discussed observations made both at vertical and oblique incidence. Mitra (1949) has analyzed the fading observed at vertical incidence both when it is due predominantly to irregular movements and also when it is due predominantly to winds. Briggs *et al.* (1950) have considered the case where fading is produced partly by a smooth translation of the ionosphere, and partly by irregular movements in it.

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APPENDIX

In § 10, equation (33), we have encountered an e.m.f. made up of Fourier components of random phase, and having a power spectrum which is a narrow function $W(f)$ centred on the frequency f_0. An e.m.f. of a similar form has also been encountered in § 7. It is the purpose of this appendix to discuss these e.m.f.'s.

E.m.f.'s of the type here described are encountered in the theory of random 'noise' and are observed if 'white' noise is passed through a narrow-band filter which passes power proportional to $W(f)$. Rice (1945) and Uhlenbeck (1945 *a*, *b*) have analyzed noise of this kind and we shall make use of their results. Since, however, the result which we require is contained only in Uhlenbeck's unpublished report, and since even there the analysis is not given in detail, we shall present it here.

The e.m.f. with which we have to deal is a quasi-periodic function $E(t)$ of t, which can be represented by

$$E(t) = \mathscr{R}[R(t)\exp\{2\pi i f_0 t + i\theta(t)\}],$$

where the amplitude $R(t)$ and the phase $\theta(t)$ vary with time slowly compared with the variation $\exp(i2\pi f_0 t)$. We first prove the following.

Theorem. Let $E(t)$ be a quasi-periodic function of t, given by

$$E(t) = \mathscr{R}[R(t)\exp\{i(\theta(t)+2\pi f_0 t)\}],$$

and let the power spectrum $W(f)$ of this function be zero except when f is near f_0, and the phase in the spectrum be random. Let $\rho_R(\tau)$ be defined by the expression

$$\rho_R(\tau) = \frac{\int_{-\infty}^{\infty} R(t)\,R(t+\tau)\,dt - \left[\int_{-\infty}^{\infty} R(t)\,dt\right]^2}{\int_{-\infty}^{\infty} [R(t)]^2\,dt - \left[\int_{-\infty}^{\infty} R(t)\,dt\right]^2}. \tag{50}$$

Then

$$\rho_R(\tau) = \frac{\left\{\left|\int_{-\infty}^{\infty} W(f_0+f)\exp(2\pi i f\tau)\,df\right|\right\}^2}{\left\{\int_{-\infty}^{\infty} W(f)\,df\right\}^2}.$$

Proof. Rice (1945) considers a function of the type $E(t)$ defined above. He denotes by R_1, R_2, θ_1, θ_2, the values of R and θ at times t_1 and t_2 separated by an interval τ. He finds (p. 77) that the joint probability distribution of R_1, R_2, θ_1, θ_2, is given by

$$p(R_1,R_2,\theta_1,\theta_2) = \frac{R_1R_2}{4\pi^2 s^2(1-z^2)}\exp\left\{-\frac{R_1^2+R_2^2-2R_1R_2 z\cos(\theta_2-\theta_1-\lambda)}{2s(1-z^2)}\right\},$$

where s is defined by

$$\int_{-\infty}^{\infty} W(f)\,df = s,$$

and z and λ are defined by

$$\int_{-\infty}^{\infty} W(f)\exp(2\pi i(f-f_0)\,\tau)\,df = sz\exp(i\lambda).$$

Now $$\overline{R_1 R_2} = \iiiint R_1 R_2 p(R_1, R_2, \theta_1, \theta_2)\, dR_1\, dR_2\, d\theta_1\, d\theta_2.$$

Put $$R_1 = A\cos\psi, \quad R_2 = A\sin\psi.$$

Then $$\frac{\partial(R_1, R_2)}{\partial(A, \psi)} = A.$$

Therefore

$$\begin{aligned}\overline{R_1 R_2} &= \iiiint \frac{A^5 \sin^2\psi \cos^2\psi}{4\pi^2 s^2(1-z^2)} \exp\left\{-A^2 \frac{1 - z\sin 2\psi \cos(\theta_2 - \theta_1 - \lambda)}{2s(1-z^2)}\right\} dA\, d\psi\, d\theta_1\, d\theta_2 \\ &= \iiint \frac{s \sin^2 2\psi (1-z^2)^2\, d\psi\, d\theta_1\, d\theta_2}{2\pi^2\{1 - z\sin 2\psi \cos(\theta_2 - \theta_1 - \lambda)\}^3} \\ &= \int_0^{\frac{1}{2}\pi} \frac{s(1-z^2)^2 \sin^2 2\psi (2 + z^2 \sin^2 2\psi)}{(1 - z^2\sin^2 2\psi)^{\frac{5}{2}}}\, d\psi \\ &= s(2E(z) - (1-z^2)\, K(z)). \end{aligned} \tag{51}$$

During these integrations we have used the following formulae:

$$\int_0^\infty A^5 \exp(-kA^2)\, dA = \frac{1}{k^3},$$

$$\int_{-\pi}^{\pi} \frac{d\phi}{(1 - b\cos\phi)^3} = \frac{\pi(2+b^2)}{(1-b^2)^{\frac{5}{2}}}, \quad \text{where} \quad |b| < 1,$$

$$\left.\begin{aligned} \int_0^{\frac{1}{2}\pi} \frac{d\chi}{\sqrt{(1 - z^2\sin^2\chi)}} &= K(z), \\ \int_0^{\frac{1}{2}\pi} \sqrt{(1 - z^2\sin^2\chi)}\, d\chi &= E(z). \end{aligned}\right\} \quad \text{These are the complete elliptic integrals.}$$

From equations (50) and (51) we obtain

$$\rho_R(\tau) = \frac{2E(z) - (1-z^2)\, K(z) - \frac{1}{2}\pi}{2 - \frac{1}{2}\pi}.$$

Now, $$E(z) = \frac{\pi}{2}\left(1 - \frac{z^2}{4} - \frac{3z^4}{64} - \frac{5z^6}{256} - \dots\right),$$

and $$K(z) = \frac{\pi}{2}\left(1 + \frac{z^2}{4} + \frac{9z^4}{64} + \frac{25z^6}{256} + \dots\right).$$

Therefore $$\rho_R(\tau) = \frac{\pi}{4(4-\pi)}\left(z^2 + \frac{z^4}{16} + \frac{z^6}{64} + \dots\right). \tag{52}$$

Now we see from the definition of z that it is proportional to the modulus of the Fourier transform of the power spectrum $W(f)$ when the frequency is shifted by an amount f_0. It is a function of τ only (it can easily be shown to be independent of f_0), and the constant of proportionality is so arranged that $z(0) = 1$; also (provided that $W(f)$ is everywhere finite) $z(\infty) = 0$. Now it follows from equation (50) that $\rho_R(0) = 1$ and $\rho_R(\infty) = 0$. Therefore $\rho_R(\tau)$ and $z(\tau)$ are equal when $\tau = 0$ or ∞. From these facts and equation (52) we deduce that the equation $\rho_R(\tau) = z^2(\tau)$ is approximately true for all values of τ. As a matter of fact the approximation is very close indeed, and the error involved in assuming that it is exactly true can be neglected for all practical purposes. We have done this throughout the foregoing paper. The approximate (almost exact) equation can be written

$$\rho_R(\tau) \doteqdot \frac{\left\{\left|\int_{-\infty}^{+\infty} W(f_0+f)\exp(2\pi i f\tau)\,df\right|\right\}^2}{\left\{\int_{-\infty}^{+\infty} W(f)\,df\right\}^2}. \tag{53}$$

The theorem represented by equation (53) is sufficient for the purposes of § 10. It is easy to prove that the right-hand side of equation (53) is independent of f_0; we can therefore replace f_0 by f_m, which is any other convenient constant frequency. If we take $f_m = -f_0$ we obtain

$$\rho_R(\tau) = \frac{\left\{\left|\int_{-\infty}^{+\infty} W(f-f_0)\exp(2\pi i f\tau)\,df\right|\right\}^2}{\left\{\int_{-\infty}^{+\infty} W(f)\,df\right\}^2}. \tag{54}$$

If we replace $W(f)$ by $\left\{\left|Q\left(\frac{\lambda(f+f_0)}{v}\right)\right|\right\}^2$ in this equation we obtain the result of equation (29).

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